6 Relations between measures

Proposition 6.1. *Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ_1, μ_2 be positive measures on $\mathcal{M}$. Then, $\mu := \mu_1 + \mu_2$ is a positive measure on $(X, \mathcal{M})$. Moreover, $\mathcal{L}^1(\mu) = \mathcal{L}^1(\mu_1) \cap \mathcal{L}^1(\mu_2)$ and*

$$\int_A f \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu_1 + \int_A f \, \mathrm{d}\mu_2 \quad \forall f \in \mathcal{L}^1(\mu), A \in \mathcal{M}.$$

Proof. **Exercise.** □

Definition 6.2 (Complex Measure)**.** Let X be a measured space with σ-algebra $\mathcal{M}$. Then, a map $\mu : \mathcal{M} \to \mathbb{C}$ is called a *complex measure* iff it is countably additive, i.e., satisfies the following property: If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{M}$ such that $A_n \cap A_m = \emptyset$ if $n \neq m$, then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Remark 6.3. 1. The above definition implies $\mu(\emptyset) = 0$. 2. The convergence of the series in the definition is absolute since its limit must be invariant under reorderings. 3. In contrast to positive measures, a complex measure is always finite.

Exercise 33. Show that the complex measures on a given σ-algebra form a complex vector space.

Definition 6.4. Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ be a positive measure on $(X, \mathcal{M})$ and ν a positive or complex measure on $(X, \mathcal{M})$. We say that ν is *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$ iff $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \in \mathcal{M}$.

Definition 6.5. Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ be a positive or complex measure on $(X, \mathcal{M})$. We say that μ is *concentrated* on $A \in \mathcal{M}$ iff $\mu(B) = \mu(B \cap A)$ for all $B \in \mathcal{M}$.

Definition 6.6. Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ, ν be positive or complex measures on $(X, \mathcal{M})$. We say that μ and ν are *mutually singular*, denoted $\mu \perp \nu$, iff there exist disjoint sets $A, B \in \mathcal{M}$ such that μ is concentrated on A and ν is concentrated on B.

Proposition 6.7. *Let μ be a positive measure and ν, ν_1, ν_2 be positive or complex measures.*

1. *If μ is concentrated on A and $\nu \ll \mu$, then ν is concentrated on A.*
2. *If $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.*

3. *If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.*

4. *If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$.*

5. *If $\nu_1 \perp \nu$ and $\nu_2 \perp \nu$, then $\nu_1 + \nu_2 \perp \nu$.*

Proof. **Exercise.** $\square$

Theorem 6.8. *Let X be a measure space with σ-algebra $\mathcal{M}$ and σ-finite measure μ. Let ν be a finite measure on $(X, \mathcal{M})$.*

1. *(Lebesgue) Then, there exists a unique decomposition*
$$\nu = \nu_a + \nu_s,$$
into finite measures such that $\nu_a \ll \mu$ and $\nu_s \perp \mu$.

2. *(Radon-Nikodym) There exists a unique $[h] \in \mathrm{L}^1(\mu)$ such that for all $A \in \mathcal{M}$,*
$$\nu_a(A) = \int_A h \, \mathrm{d}\mu.$$

Proof. We first show the uniqueness of the decomposition $\nu = \nu_a + \nu_s$ in (1.). Suppose there is another decomposition $\nu = \nu'_a + \nu'_s$. Note that all the measures involved here are finite and thus are also complex measures. In particular, we obtain the following equality of complex measures, $\nu_a - \nu'_a = \nu'_s - \nu_s$. However, by Proposition 6.7 the left hand side is absolutely continuous with respect to μ while the right hand side is singular with respect to μ. Again by Proposition 6.7, the equality of both sides implies that they must be zero, i.e., $\nu'_a = \nu_a$ and $\nu'_s = \nu_s$.

To show the uniqueness of $[h] \in \mathrm{L}^1(\mu)$ in (2.) we note that given another element $[h'] \in \mathrm{L}^1(\mu)$ with the same property, we would get $\int_A (h - h') \, \mathrm{d}\mu = 0$ for all measurable sets A. By Proposition 3.21 then $0 = [h - h'] = [h] - [h'] \in \mathrm{L}^1(\mu)$.

We proceed to construct the decomposition $\nu = \nu_a + \nu_s$ and the element $[h] \in \mathrm{L}^1(\mu)$. By Lemma 4.21, there is a function $w \in \mathcal{L}^1(\mu)$ with $0 < w < 1$. This yields the finite measure μ_w, given by
$$\mu_w(A) := \int_A w \, \mathrm{d}\mu \quad \forall A \in \mathcal{M}.$$
(Recall the last part of Exercise 28.) Define the finite measure $\varphi := \nu + \mu_w$. Note that $\mathcal{L}^1(\varphi) \subseteq \mathcal{L}^1(\nu)$ and $\mathcal{L}^1(\varphi) \subseteq \mathcal{L}^1(\mu_w)$ and we have (using Proposition 6.1),
$$\int_X f \, \mathrm{d}\varphi = \int_X f \, \mathrm{d}\nu + \int_X f w \, \mathrm{d}\mu \quad \forall f \in \mathcal{L}^1(\varphi). \tag{1}$$

In particular, we may deduce

$$\left|\int_X f\mathrm{d}\nu\right| \le \|f\|_{\nu,1} \le \|f\|_{\varphi,1} \quad \forall f \in \mathcal{L}^1(\varphi).$$

By Proposition 4.20 we have $\mathcal{L}^2(\varphi) \subseteq \mathcal{L}^1(\varphi)$ and even

$$\|f\|_{\varphi,1} \le \|f\|_{\varphi,2}\,(\varphi(X))^{1/2} \quad \forall f \in \mathcal{L}^2(\varphi).$$

Combining the inequalities we find

$$\left|\int_X f\mathrm{d}\nu\right| \le \|f\|_{\varphi,2}\,(\varphi(X))^{1/2} \quad \forall f \in \mathcal{L}^2(\varphi).$$

This means that the linear map $\alpha : \mathrm{L}^2(\varphi) \to \mathbb{K} \subseteq \mathbb{C}$ given by $[f] \mapsto \int_X [f]\mathrm{d}\nu$ is bounded. Since $\mathrm{L}^2(\varphi)$ is a Hilbert space, Theorem 4.28 implies that there is an element $g \in \mathcal{L}^2(\varphi)$ such that $\alpha([f]) = \langle [f], [\overline{g}]\rangle$ for all $f \in \mathcal{L}^2(\varphi)$. This implies,

$$\int_X f\mathrm{d}\nu = \int_X fg\,\mathrm{d}\varphi \quad \forall f \in \mathcal{L}^2(\varphi) \tag{2}$$

By inserting characteristic functions for f we obtain

$$\nu(A) = \int_A g\,\mathrm{d}\varphi \quad \forall A \in \mathcal{M}.$$

On the other hand we have $\nu(A) \le \varphi(A)$ for all measurable sets A and hence,

$$0 \le \frac{1}{\varphi(A)}\int_A g\,\mathrm{d}\varphi = \frac{\nu(A)}{\varphi(A)} \le 1 \quad \forall A \in \mathcal{M} : \varphi(A) > 0.$$

We can now apply the Averaging Theorem (Theorem 3.20) to conclude that $0 \le g \le 1$ almost everywhere. We modify g on a set of measure zero if necessary so that $0 \le g \le 1$ everywhere. In particular, if $f \in \mathcal{L}^2(\varphi)$ then $(1-g)f \in \mathcal{L}^2(\varphi)$ and $gf \in \mathcal{L}^2(\varphi)$. Combining (1) and (2) we find

$$\int_X (1-g)f\,\mathrm{d}\nu = \int_X fgw\,\mathrm{d}\mu \quad \forall f \in \mathcal{L}^2(\varphi).$$

Set $Z_a := \{x \in X : g(x) < 1\}$ and $Z_s := \{x \in X : g(x) = 1\}$ and define the measures $\nu_a(A) := \nu(A \cap Z_a)$ and $\nu_s := \nu(A \cap Z_s)$ for all $A \in \mathcal{M}$. Since X is the disjoint union of Z_a and Z_s we obviously have $\nu = \nu_a + \nu_s$. Taking f to be the characteristic function of Z_s we find that $\int_{Z_s} w\,\mathrm{d}\mu = 0$. Since $0 < w$, we conclude that $\mu(Z_s) = 0$. In particular, this implies that μ is supported on Z_a, while ν_s is supported on Z_s, so $\nu_s \perp \mu$.

Define now the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n := \sum_{k=1}^n g^{k-1}$. Since g is bounded, f_n is bounded. Multiplying with characteristic functions we find for measurable sets A,

$$\int_A (1-g^n)\,\mathrm{d}\nu = \int_A (1-g)f_n\,\mathrm{d}\nu = \int_A f_n g w\,\mathrm{d}\mu.$$

Note that $\{1-g^n\}_{n\in\mathbb{N}}$ increases monotonically and converges pointwise to the characteristic function of Z_a. Thus, by the Monotone Convergence Theorem (Theorem 3.26) or by the Dominated Convergence Theorem (Theorem 3.29) the left hand side converges to $\nu(A \cap Z_a) = \nu_a(A)$.

The sequence $\{f_n g w\}_{n\in\mathbb{N}}$ is also increasing monotonically with its μ-integrals over A bounded by $\nu_a(A)$. So the Monotone Convergence Theorem (Theorem 3.26) applies and the pointwise limit is a μ-integrable function h. We get

$$\nu_a(A) = \int_A h \, \mathrm{d}\mu,$$

showing existence in (2.) and also $\nu_a \ll \mu$, thus completing the existence proof for (1.). $\square$

Remark 6.9. The function h appearing in the above Theorem is also called the *Radon-Nikodym derivative*, denoted as $h = \mathrm{d}\nu_a/\mathrm{d}\mu$.